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PROBLEMS OF INTENSE MAGNETIC FIELD IN GRAVITATIONAL COLLAPSE

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Recently great interest has been directed toward problems of gravitational collapse. As the theory of collapse becomes more advanced, complications (such as rotation and magnetic field), which were neglected in pioneering work, are being gradually included. As pointed out by Woltjer (1964) some years ago, field strengths of the order of 10^{14} gauss may be achieved in gravitational collapse. The classical theory of electrodynamics is expected to break down when the spin interaction energy $\mu_B H$ (where μ_B is the Bohr magneton $= e\hbar/2mc$ and H is the magnetic field) exceeds mc^2; it would therefore be possible, according to classical concepts of energy conservation, to create a pair of electrons spontaneously with proper orientations of spin when the field strength exceeds H_q (where $H_q = m^2c^3/e\hbar = 4.414 \times 10^{13}$ gauss). Questions have been raised as to whether a magnetic field may be destroyed by spontaneous pair creation. To answer this question and others, it is necessary to develop a quantum theory of matter in intense magnetic fields.

In a series of papers (Chiu and Canuto 1968; Canuto and Chiu 1968*a–c*), we have studied the detailed properties of matter in intense magnetic fields. In this paper we shall discuss some results of astrophysical interest. We have considered two cases: (*a*) the magnetic moment of the electron is the Dirac moment $e\hbar/2mc$ and (*b*) in addition to the Dirac moment the electron also possesses a Schwinger anomalous magnetic moment $\alpha e\hbar/4\pi mc$. When the field is of the order of 10^{16} gauss, the anomalous magnetic moment becomes important and must be taken into account (Chiu, Canuto, and Fassio-Canuto 1968).

a) According to Dirac's theory the magnetic moment of the electron is exactly 1 Bohr magneton, μ_B. Solutions to the Dirac equation of an electron in an external magnetic field were obtained several decades ago (Rabi 1928). The energy eigenvalues are (Canuto and Chiu 1968*a*)

$$E = \pm mc^2[1 + x^2 + (2n + 1 + s)H/H_q]^{1/2} , \tag{1}$$

where n, the principal quantum number characterizing the size of the circular motion of an electron in a magnetic field, takes values from 0 to ∞; $s = \pm 1$ characterizes the two spin states, where $s = -1$ corresponds to the parallel and $s = +1$ to the antiparallel cases. H is the magnetic field in the z-direction ($H_q = m^2c^3/e\hbar = 4.414 \times 10^{13}$ gauss); $x = p_z/mc$; p_z is the z-momentum of the electron; m is electron mass; and the rest of the symbols have their usual meanings. The plus and minus signs correspond to electron and positron states, respectively. It is seen in equation (1) that there is a twofold degeneracy between the state n and $s = 1$ and the state $n + 1$ and $s = -1$. The lowest-

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energy states of the electron and positron ($x = 0$, $n = 0$, $s = -1$) are $\pm mc^2$. This means that the separation energy between the positron and electron states is still $2mc^2$, unaltered by the presence of a magnetic field. This implies that pairs are not created spontaneously at field strengths greater than 10^{14} gauss even when the classical spin interaction energy $\mu_B H$ exceeds mc^2.

The equations of state, however, exhibit a strong anisotropy, especially when the electrons are mainly in states of small quantum numbers n (Canuto and Chiu 1968*a*, *b*). The most important case of interest is that of a degenerate electron gas in a magnetic field. In this case, states up to some quantum number m and some Fermi energy E_F (E_F includes the rest energy mc^2) are occupied. The equations of state are

$$P_{xx} = P_{yy} = \pi^{-2}(mc^2/\lambda\!\!\!^{-}_c{}^3)(H/H_q)^2 \sum_{n=1}^{m} nC_1(\mu/a_n) , \tag{2}$$

$$P_{zz} = \pi^{-2}(mc^2/\lambda\!\!\!^{-}_c{}^3)(H/H_q)\left[\tfrac{1}{2}C_2(\mu) + \sum_{n=1}^{m} a_n{}^2 C_2(\mu/a_n)\right] , \tag{3}$$

$$U = \pi^{-2}(mc^2/\lambda\!\!\!^{-}_c{}^3)(H/H_q)\left[\tfrac{1}{2}C_3(\mu) + \sum_{n=1}^{m} a_n{}^2 C_3(\mu/a_n)\right] , \tag{4}$$

$$N = \pi^{-2}\lambda\!\!\!^{-}_c{}^{-3}(H/H_q)\left[\tfrac{1}{2}C_4(\mu) + \sum_{n=1}^{m} a_n C_4(\mu/a_n)\right] , \tag{5}$$

where P_{xx}, P_{yy}, and P_{zz} are normal stresses in the x-, y-, and z-directions; U is the total energy density (including the rest energy of the electrons); N is the particle number density; $\mu = E_F/mc^2$; $a_n = (1 + 2nH/H_q)^{1/2}$; and $\lambda\!\!\!^{-}_c = \hbar/mc$ is the Compton wavelength of the electron. The functions $C_k(x)$ are defined as

$$C_1(\mu) = \ln\left[\mu^2 + (\mu^2 - 1)^{1/2}\right] , \quad C_3(\mu) = \tfrac{1}{2}C_1(\mu) + \tfrac{1}{2}\mu(\mu^2 - 1)^{1/2} , \tag{6}$$

and

$$C_4(\mu) = (\mu^2 - 1)^{1/2} , \quad C_2(\mu) = C_3(\mu) - C_2(\mu) , \tag{7}$$

and the upper limit of summation m is determined by the condition

$$a_m \leq \mu < a_{m+1} . \tag{8}$$

The pressure of the gas is very anisotropic, and, for values of μ such that $\mu < a_1 = (1 + 2H/H_q)^{1/2}$, the stress perpendicular to the field (P_{xx} and P_{yy}) even vanishes and the gas in this limit is exactly a one-dimensional gas! Figure 1 shows the behavior of the equation of state. In general, all thermodynamic variables have kinks and discontinuous derivatives where a new magnetic state is excited. At large values of m, the equation of state approaches that of a degenerate electron gas obtained by Chandrasekhar (1967).

The total induced magnetic moment M of an electron gas is given by the following expression (Canuto and Chiu 1968*c*)

$$M = \mathfrak{M}/\mathfrak{M}_0 = \tfrac{1}{2}C_2(\mu) + \sum_{n=1}^{m} a_n{}^2 C_2(\mu/a_n) - (H/H_q)\sum_{n=1}^{m} nC_1(\mu/a_n) . \tag{9}$$

In general, the maximum value of induced field due to induced magnetic moment is only 10^{-3} of that of the impressed field. On the basis of this result, we concluded that ferro-

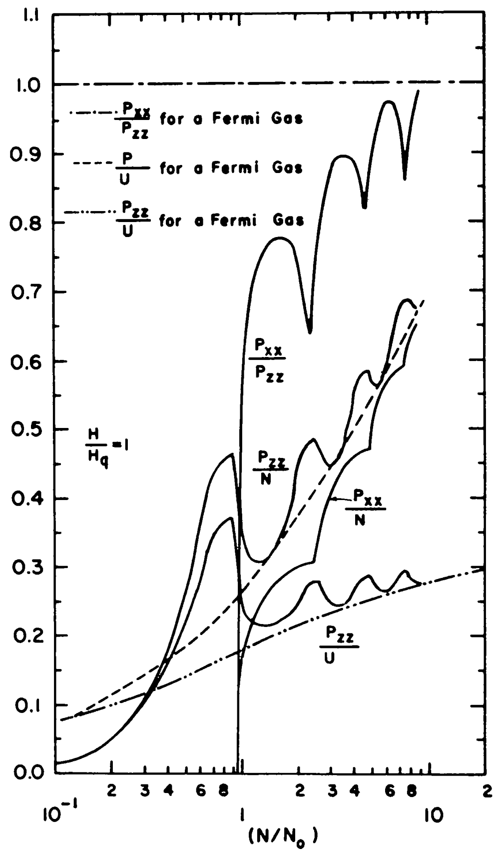

FIG. 1.—Functional dependence of P_{xx}/P_{zz}, P_{zz}/N, P_{xx}/N, and P_{zz}/N versus N/N_0 ($N_0 = \pi^{-2}\lambda_c^{-3}$) for the degenerate case with $H/H_q = 1$. Corresponding functions for a Fermi gas are also shown for comparison.

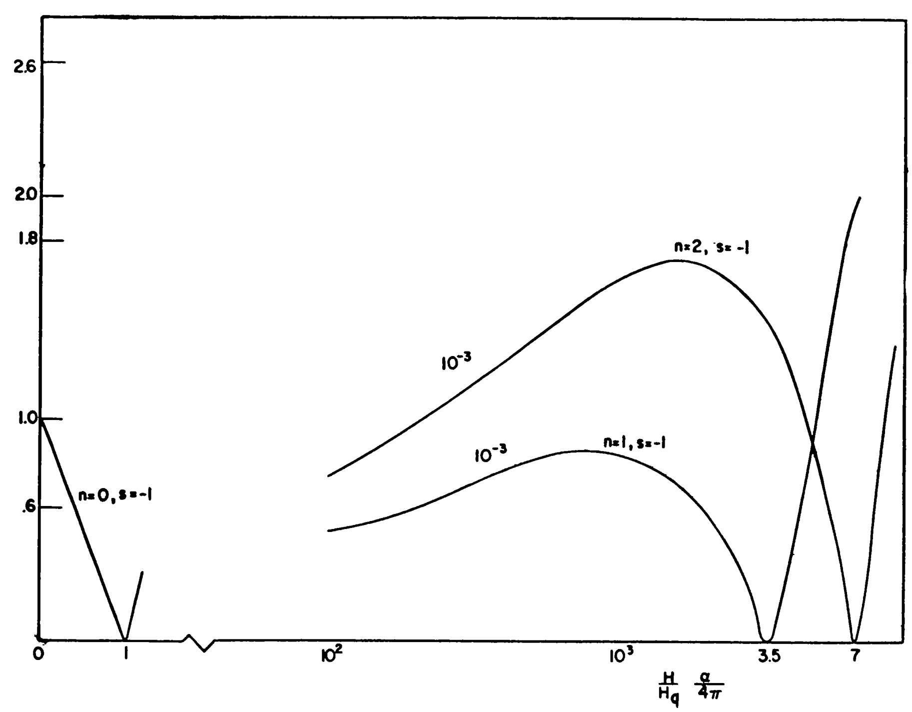

FIG. 2.—Energy eigenvalues (in units of mc^2) for the cases $n = 0$, $s = -1$; $n = 1$, $s = -1$; and $n = 2$, $s = -1$ as a function of H/H_c ($H_c = 4\pi H_q/a$). The case $x = 0$ is considered.

magnetism will not exist in dense electron gas. The presence of magnetic fields in collapsed bodies (if any) must therefore be due to macroscopic currents.

b) The electron possesses an anomalous magnetic moment of the amount $\alpha e\hbar/4\pi mc$ ($\alpha = \frac{1}{137}$) in addition to its Dirac moment, $e\hbar/2mc$ (Schwinger 1948). Solutions to the Dirac equation with an anomalous magnetic moment have recently been obtained by Ternov, Bagzov, and Zhukovskii (1966). They give the following energy states:

$$E = \pm mc^2\langle x^2 + \{[1 + (2n + 1 + s)H/H_q]^{1/2} + s\alpha H/4\pi H_q\}^2\rangle^{1/2} , \tag{10}$$

where all variables have the same meaning as in equation (1), to which it reduces when we set $\alpha = 0$. The inclusion of the anomalous magnetic moment removes the twofold degeneracy of equation (9). The quantization of electron energy and the removal of degeneracy are very similar to Zeeman splitting of the atomic spectrum. Figure 2 shows the energy eigenvalues as a function of the field strength $\theta = H/H_c$, where $H_c = 4\pi H_q/\alpha$.

According to equation (10) the lowest-energy states of the electron or positron (with $x = 0, s = -1$, and arbitrary n) are zero when the field strength H satisfies the following conditions:

$$1 + 2nH/H_q = (\alpha H/4\pi H_q)^2 \tag{11}$$

or

$$H/H_q = (4\pi/\alpha)^2\{n + [n^2 + (\alpha/4\pi)^2]^{1/2}\} \rightarrow \begin{cases} 4\pi/\alpha \simeq 10^3 & (n = 0) , \\ \simeq 2n(4\pi/\alpha)^2 & (n \geq 1) . \end{cases} \tag{12}$$

If it were possible for the electron to possess a negative energy, spontaneous pair creation would be possible, at the expense of the energy of the magnetic field. However, the sign before the square root in the expression for the energy, equation (10), is an invariant property of the electron, so that the energy of an electron can never become less than zero (see Fig. 2). This means that the energy of the electron will never cross that of the positron (non-crossing property).

We thus conclude that spontaneous pair creation will not take place at all at the expense of magnetic-field energy, even when the anomalous magnetic moment is taken into account.

Electron pairs can still be created, however, at the expense of the thermodynamic energy of the system. When the field strength approximately satisfies equation (12), the rest mass of the electron is small and pair creation can take place even at temperatures that are small compared to $mc^2/k = 6 \times 10^9$ ° K. Expressions for the pair density have been given previously (Chiu, Canuto, and Fassio-Canuto 1968). An interesting case occurs when

$$mc^2 \gg kT \gg mc^2|(1 + 2nH/H_q)^{1/2} - \alpha H/4\pi H_q| , \tag{13}$$

i.e., when $T \ll 6 \times 10^9$ ° K and when the field strength is sufficiently close to those given by equation (12). In this case pair creation is negligible in all states except the one that satisfies equation (12). We find

$$n_+ = N_0(kT/mc^2)\ln[1 + \exp(-mc^2\mu_c/kT)] , \quad n_- = n_0 + n_+ , \tag{14}$$

where n_+, n_- are positron and electron number densities; $N_0 = \pi^{-2}\lambda_c^{-3}$; n_0 is the number density of electrons without pair creation; and μ_c is the chemical potential of the electron (in units of mc^2) for the state m, including the equivalent rest energy for the state μ_m, which is

$$\mu_m = (1 + 2nH/H_q)^{1/2} - 4\pi H/\alpha H_q \tag{15}$$

($\mu_m \simeq 0$ according to our assumption). In vacuum $\mu_c = 0$, and

$$n_+ = n_- = N_0(kT/mc^2) \ln 2 , \tag{16}$$

independent of the identity of the state that satisfies equation (12).

The pair density thus vanishes when $T = 0$, in accordance with our conclusion shown earlier. However, processes such as $e^- + e^+ \rightarrow \nu + \bar{\nu}$ can still take place, dissipating the energy of the system. This problem is currently under investigation.

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